COUNTING MANIFOLDS WITH BOUNDED ASYMPTOTIC INVARIANTS

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ABSTRACT. It is known that there are only finitely many compact Riemannian manifolds with bounded embolic volume. That is, if we consider the class of manifolds with volume bounded above and injectivity radius bounded below, then there are only finitely many manifolds within this class. Morevoer, Sabourau proved that there are finitely many hyperbolic manifolds with bounded systolic volume. We establish quantitative versions of these finiteness theorems.

CONTENTS

1. INTRODUCTION

The well known systolic inequality of Gromov says that homotopy 1-systole of a Riemannian manifold is bounded above in terms of volume. Moreover, Gromov proved that the optimal constant, called systolic volume in this paper, is a topological invariant representing the topological complicatedness of the manifold. In Berger's embolic inequality, the optimal constant is analogous to systolic volume, also representing the topological complicatedness.

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We study quantitative descriptions of manifolds with bounded systolic volume or bounded embolic volume in this paper.

In differential geometry, Yamaguchi [Yam88] proved that there are only finitely many homotopy types for compact manifolds with bounded embolic volume. Moreover, Grove, Petersen and Wu [GPW90] showed that homeomorphism types are also finite if the embolic volume is bounded. In [Sab07], Sabourau proved that hyperbolic manifolds with bounded systolic volume is finite. In contrast, it is well known that hyperbolic manifolds with dimension at least four is finite when the volume is bounded. There are finitely many hyperbolic 3-manifolds, if volume is bounded from above and the injectivity radius is bounded from below. Moreover, in Burger, Gelander, Lubotzky and Mozer [BGLM02], a precise quantitative version of Wang's theorem is established. In this paper, we study the problem of counting manifolds with bounded embolic volume or bounded systolic volume. Analogous to [BGLM02], our work is a quantitative version to the finiteness theorems with bounded embolic volume or bounded systolic volume.

Counting homotopy types of manifolds. Quantitative study of homotopy types of compact manifolds can be traced back to Mather [Mat65] and Kister [Kis68]. They proved that homotopy types of compact manifolds are countable. Weinstein [Wei67] showed finiteness of homotopy types with curvature constraints. A complete geometric description of homotopy finiteness is given by Grove and Petersen [GP88]. Furthermore, Yamaguchi [Yam88] showed the finiteness of homotopy types of compact manifolds with bounded embolic volume. Our first result in this paper is a quantitative version of Yamaguchi's theorem.

Let M be a compact n-dimensional manifold endowed with a Riemannian metric $\mathcal{G}$, denoted $(M, \mathcal{G})$. Denote by $\mathsf{Inj}(M, \mathcal{G})$ the injectivity radius of $(M, \mathcal{G})$. The embolic volume of M, denoted $\mathsf{Emb}(M)$, is defined to be

$$\inf_{\mathcal{G}} \frac{\mathsf{Vol}_{\mathcal{G}}(M)}{\mathsf{Inj}(M, \mathcal{G})^n},$$

where the infimum is taken over all Riemannian metrics $\mathcal{G}$ on M. According to Berger's embolic inequality (see [Ber03, Section 7.2.4])

$$\mathsf{Inj}(M, \mathcal{G})^n \leqslant C_n \, \mathsf{Vol}_{\mathcal{G}}(M), \tag{1.1}$$

any compact manifold M of dimension n has a positive embolic volume.

Theorem 1.1 (Yamaguchi [Yam88])**.** *For any positive constant L, there are only finitely many homotopy types of compact manifolds M with $\mathsf{Emb}(M) \leqslant L$.*

The first result of this paper is the following quantitative version of the above Yamaguchi's theorem.

Theorem 1.2. *Let $\varphi_n(L)$ be the number of homotopy types of compact aspherical manifolds of dimension n, with $\mathsf{Emb}(M) \leqslant L$. We have*

$$a_n L^{\frac{n+1}{n(n-1)}} \log L \leqslant \log \varphi_n(L) \leqslant b_n \tilde{L}^3 \log \tilde{L}, \tag{1.2}$$

where a_n, b_n are two positive constants only depending on n, $\tilde{L}$ is a positive constant only depending on L.

Remark 1.3. The constant $\tilde{L}$ is taken as

$$\sup\{\mathsf{CV}(M) | \, \mathsf{Emb}(M) \leqslant L\},$$

where the supremum is taken over all compact aspherical n-manifolds M with $\mathsf{Emb}(M) \leqslant L$, see the proof of this theorem in the following.

Let convexity radius of $(M,\mathcal{G})$ be denoted by $\mathsf{Conv}(M,\mathcal{G})$. Note that $2\,\mathsf{Conv}(M,\mathcal{G}) \leqslant \mathsf{Inj}(M,\mathcal{G})$,

$$\mathsf{Conv}(M,\mathcal{G})^n \leqslant C_n\,\mathsf{Vol}_{\mathcal{G}}(M) \tag{1.3}$$

also holds for any compact manifold M of dimension n. Analogous to embolic volume, we define convex volume of a compact n-dimensional manifold M to be

$$\mathsf{CV}(M) = \inf_{\mathcal{G}} \frac{\mathsf{Vol}_{\mathcal{G}}(M)}{\mathsf{Conv}(M,\mathcal{G})^n},$$

where the infimum is taken over all Riemannian metrics $\mathcal{G}$ on M. For a sufficiently large positive number C, denote by $\theta_n(C)$ the number of homotopy types of compact aspherical manifolds of dimension n, with $\mathsf{CV}(M) \leqslant C$. Theorem 1.2 is proved by the following counting estimate with $\mathsf{CV}(M)$.

Theorem 1.4. *Let C be a sufficiently large positive number. Then there exist positive constants c_n and $\tilde{c}_n$, which only depend on n, such that*

$$c_n\,C^{\frac{n+1}{n(n-1)}}\log C \leqslant \log\theta_n(C) \leqslant \tilde{c}_n\,C^3\log C. \tag{1.4}$$

Homeomorphism type of aspherical manifolds. After Mather's work on homotopy types, Cheeger and Kister [CK70] showed that homeomorphism types of compact manifolds are also countable. Moreover, Cheeger's finiteness theorem implies finiteness of homeomorphism types of Riemannian manifolds with geometric constraints. Grove, Petersen and Wu [GPW90, GPW91] proved finiteness of homeomorphism types of manifolds with bounded embolic volume, which is a curvature free finiteness theorem. In this paper, we give a quantitative version of the theorem in [GPW90].

The method used in the proof of Theorem 1.2 and Theorem 1.4 is based on counting in terms of fundamental groups. Therefore, for manifolds whose homeomorphism types are determined by fundamental groups, we get counting estimates of homeomorphism types. A manifold M is topologically rigid, if any homotopy equivalence $f: N \to M$ from a manifold N to M is homotopic to a homeomorphism. Homeomorphism types of topogically rigid aspherical manifolds are determined by fundamental groups.

Theorem 1.5. *Let M be a compact aspherical manifold with homeomorphism type determined by its fundamental group. For a sufficiently large positive constant C, denote by $\mu_n(C)$ the number of homeomorphism types of M with $\mathsf{Emb}(M) \leqslant C$, and by $\nu_n(C)$ the number of homeomorphism types of M with $\mathsf{CV}(M) \leqslant C$. Then we have*

(1)
$$f_n\,C^{\frac{n+1}{n(n-1)}}\log C \leqslant \log\mu_n(C) \leqslant g_n\,\tilde{C}^3\log\tilde{C},$$
where f_n and g_n are two positive constants only depending on n, and $\tilde{C}$ is a positive constant only depending on C.

(2)
$$k_n C^{\frac{n+1}{n(n-1)}}\log C \leqslant \log\nu_n(C) \leqslant l_n C^3\log C,$$
where k_n and l_n are two positive constants only depending on n.

Remark 1.6. Borel's conjecture asserts that fundamental group of aspherical manifolds determines homeomorphism type. Hence if Borel's conjecture becomes true, Theorem 1.5 holds for all compact aspherical manifolds.

Remark 1.7. A complete Riemannian manifold M has normalized bounded negative curvature, if sectional curvature of M lies in a closed subinterval of $[-1, 0)$. For complete Riemannian manifold of dimension $n \geqslant 5$ and with normalized bounded negative curvature, assume that volume is at least v, in [BGS20] it is shown that the number $\mathcal{P}_n(v)$ of homeomorphism types satisfies

$$\alpha\, v \log v \leqslant \log \mathcal{P}_n(v) \leqslant \beta\, v \log v,$$

where α and β are two positive constants, $n \geqslant 5$. This result is based on Farrell and Jones's work [FJ98] of Borel conjecture. If we apply Theorem 1.2 and Theorem 1.4, then we get counting results with embolic volume and convex volume for manifolds with a complete Riemannian structure of normalized bounded negative curvature.

Counting hyperbolic manifolds. For complete hyperbolic manifolds of dimension at least 4, Wang's theorem implies that there are finitely many of them with bounded volume, see [Wan72]. In [BGLM02], Burger and Gelander et al. proved a quantitative version of Wang's theorem.

Let M be a closed n-dimensional manifold endowed with Riemannian metric $\mathcal{G}$, denoted $(M, \mathcal{G})$. The systole of $(M, \mathcal{G})$, denoted $\mathsf{Sys}\,\pi_1(M, \mathcal{G})$, is defined to be the shortest length of a non contractible loop in M. Then the systolic volume of a closed manifold M of dimension n, denoted $\mathsf{SR}(M)$, is defined to be

$$\inf_{\mathcal{G}} \frac{\mathsf{Vol}_{\mathcal{G}}(M)}{\mathsf{Sys}\,\pi_1(M, \mathcal{G})^n},$$

where the infimum is taken over all Riemannian metrics $\mathcal{G}$ on M.

Combining Gromov's theorem on systolic volume (see Theorem 2.4 in the following) and Burger and Gelander et al.'s theorem , we know that the finiteness theorem holds for hyperbolic n-manifolds ($n \geqslant 4$) with bounded systolic volume. Moreover, in [Sab07] Sabourau showed that the finiteness also holds for hyperbolic 3-manifolds with bounded systolic volume. In this paper, we show a quantitative version of the finiteness theorem with bounded systolic volume by using the method of Burger, Gelander et al. in [BGLM02]. This result can be obtained directly from the method of [BGLM02] and Gromov's theorem in systolic geometry. We further study the problem of counting hyperbolic manifold by using embolic volume.

Theorem 1.8. *Let $n \geqslant 3$, and $C > 0$ be a sufficiently large positive number. Denote by $\rho_n(C)$ the number of closed hyperbolic n-manifolds M with $\mathsf{SR}(M) \leqslant C$. Then we have*

$$a'_n C \log^{n+1} C \leqslant \log \rho_n(C) \leqslant b'_n C \log^{n+1} C$$

if $n \geqslant 4$, where a'_n and b'_n are positive constants only depending on n. In dimension 3, we have

$$\mathbf{a}\, C \log C \leqslant \log \rho_3(C) \leqslant A(C)\, C \log^4 C,$$

where $\mathbf{a}$ is a fixed positive constant, and $A(C)$ a function only depending on C.

Remark 1.9. In the above theorem, the function $A_2(C)$ of C serves as lower bound of injectivity radius for closed hyperbolic 3-manifolds M with $\mathsf{SR}(M) \leqslant C$.

Theorem 1.8 is proved by using the method in [BGLM02] and Gromov's theorem of connecting simplicial volume with systolic volume.

Mostow's rigidity theorem implies isometry type of compact hyperbolic manifolds are determined by fundamental group. Therefore, Theorem 1.5 can be applied to have the following counting estimate.

Proposition 1.10. *Assume that $n \geqslant 3$, then for any sufficiently large positive constant C, denote by $\rho'_n(C)$ the number of compact hyperbolic manifolds M with $\mathsf{Emb}(M) \leqslant C$.*

(1) *If $n \geqslant 4$, we have*

$$a''_n C \log^{n+1} C \leqslant \log \rho'_n(C) \leqslant b''_n C \log^{n+1} C, \tag{1.5}$$

where a''_n and b''_n are two positive constant only depending on n.

(2) *If $n = 3$, we have*

$$bC \log C \leqslant \log \rho'_3(C) \leqslant B(C)\, C \log^4 C, \tag{1.6}$$

where b is a fixed constant, and $B(C)$ a function only related to C.

Remark 1.11. By a check of the proof of Theorem 1.5, we know analogous estimates of Proposition 1.10 also holds for compact hyperbolic manifolds M with bounded $\mathsf{CV}(M)$.

Homotopy complexity and covering type. The counting problem is related to homotopy complexity. A (d, v)-simplicial complex is a simplicial complex with at most v vertices, and degree of each vertex is at most d. A family $\mathcal{F}$ of d-dimensional Riemannian manifolds has uniform homotopy complexity, if each $M \in \mathcal{F}$ is homotopy equivalent to some $(D, \alpha \cdot \mathsf{Vol}_{\mathcal{G}}(M))$-simplicial complex, where D, α are some fixed positive constants. Uniform homotopy complexity of locally symmetric Riemannian manifolds is studied in [Gel04]. Recent progresses can be found in [Frą21, GV21].

A topological uniform homotopy complexity theorem is given in the following.

Theorem 1.12. *Let M be a compact manifold of dimension n. Then M is homotopy equivalent to a $(A_n \cdot \mathsf{CV}(M), A_n \cdot \mathsf{CV}(M))$-simplicial complex, with A_n a positive constant only depending on n.*

Remark 1.13. Theorem 1.12 can be seen as a curvature free generalization to Weinstein's geometric finiteness theorem (see [Wei67]).

A topological invariant called covering type is defined in [KW16]. For compact manifolds M, the covering type is defined to be the minimum number of contractible subsets in an open cover of any space homotopy equivalent to M. Theorem 1.12 has an application on the covering type.

Corollary 1.14. *The covering type $\mathsf{ct}(M)$ of a compact n-dimensional manifold M satisfies*

$$\mathsf{ct}(M) \leqslant \delta_n\, \mathsf{CV}(M), \tag{1.7}$$

where δ_n is a positive constant only depending on n.

Organization. This paper is organized as follows: We introduce some background knowledge in Section 2. In Section 3, we briefly review progresses on counting hyperbolic manifolds. The estimate of the number of homotopy types of compact manifolds is given in Section 4. In section 5, we consider counting hyperbolic manifolds in terms of systolic volume and embolic volume. In section 6, homotopy complexity of compact manifolds M is studied in terms of convex volume $\mathsf{CV}(M)$.

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2. Preliminaries

2.1. **Nerve of good cover.** Given a topological space X and a covering $\mathcal{C}$ on X, the nerve $\mathcal{N}$ associated to X is a simplicial complex, such that for each subset in $\mathcal{C}$ there is a corresponding vertex in $\mathcal{N}$, and for each nonempty intersection of subsets in $\mathcal{C}$, there is a simplex in $\mathcal{N}$ spanned by the corresponding vertices.

Definition 2.1. An open covering $\mathcal{C}$ of X is called good cover, if any nonempty intersection of subsets in $\mathcal{C}$ is contractible.

Theorem 2.2 (Nerve lemma, see Hatcher [Hat02, Corollary 4G.3.])**.** *If $\mathcal{C}$ is a good cover of a topological space X, then the nerve Y associated to $\mathcal{C}$ is homotopy equivalent to X.*

2.2. **Hyperbolic volume and simplicial volume.** A closed manifold M of dimension n is hyperbolic if it admits a complete Riemannian metric of constant sectional curvature -1. Denote by hyp the hyperbolic metric on M, and by $\mathsf{Vol}_{\mathsf{hyp}}(M)$ the hyperbolic volume of M. The simplicial volume of M, denoted $\|M\|$, is defined in terms of generator of homology group $H_n(M;\mathbb{R})$ with real coefficients. Gromov and Thurston showed that the simplicial volume is a topological description of hyperbolic volume, see [BP92] for reference.

Theorem 2.3 (Gromov, Thurston)**.** *If M is an oriented compact hyperbolic manifold then*

$$\|M\| = \frac{\mathsf{Vol}_{\mathsf{hyp}}(M)}{\mathcal{V}_n},$$

where $\mathcal{V}_n$ is maximal volume of an ideal n-simplex in the hyperbolic space $\mathbb{H}^n$.

2.3. **Topological complexity and systolic volume.** A central theorem in systolic geometry is Gromov's work of relating systolic volume to simplicial volume.

Theorem 2.4 (Gromov 1983, [Gro83, Theorem 6.4.D'])**.** *Let M be a closed manifold of dimension n with nonzero simplicial volume. There exist constants C_n and C'_n only depending on n, such that*

$$\|M\| \leqslant C_n\, \mathsf{SR}(M) \log^n \big(C'_n\, \mathsf{SR}(M)\big). \tag{2.1}$$

Gromov's theorem (Theorem 2.4) indicates that systolic volume is a topological invariant representing how topologically complicated an essential manifold is.

Remark 2.5. In [Gro83, Theorem 6.4.D'], there is a typo of missing n on the logarithm. The proof of Theorem 2.4 uses smoothing technique. We refer to [BK19] and [Che22] for more details about the smoothing.

Moreover, there are other geometric invariants analogous to systole: filling radius $\mathsf{FillRad}(M,\mathcal{G})$, injectivity radius $\mathsf{Inj}(M,\mathcal{G})$, convex radius $\mathsf{Conv}(M,\mathcal{G})$. Define the invariant $\mathsf{FR}(M)$ of M to be

$$\inf_{\mathcal{G}} \frac{\mathsf{Vol}_{\mathcal{G}}(M)}{\mathsf{FillRad}(M,\mathcal{G})^n},$$

where the infimum is taken over all Riemannian metrics $\mathcal{G}$ on M. Then we have the following topological invariants defined on a manifold M: $\mathsf{FR}(M), \mathsf{SR}(M), \mathsf{Emb}(M), \mathsf{CV}(M)$. Since we have the following comparison relation,

$$4\,\mathsf{Conv}(M,\mathcal{G}) \leqslant 2\,\mathsf{Inj}(M,\mathcal{G}) \leqslant \mathsf{Sys}\,\pi_1(M,\mathcal{G}) \leqslant 6\,\mathsf{FillRad}(M,\mathcal{G}),$$

the related above four invariants satisfy

$$\mathsf{CV}(M) \geqslant \mathsf{Emb}(M) \geqslant \mathsf{SR}(M) \geqslant \frac{1}{6^n}\,\mathsf{FR}(M).$$

It is proved by Brunnbauer [Bru08] that $\mathsf{FR}(M)$ does not relate to the topology of M, which only depends on the dimension of M. Gromov's work indicates that the left three invariants are all related to topological complexity of M. The relation between systolic volume $\mathsf{SR}(M)$ and other topological invariants is studied by Gromov and other people. The relation between embolic volume $\mathsf{Emb}(M)$ and topology is investigated in [Ber03] and [Che19]. In this paper, we consider counting problems in terms of embolic volume $\mathsf{Emb}(M)$, systolic volume $\mathsf{SR}(M)$ and convex volume $\mathsf{CV}(M)$.

3. Counting hyperbolic manifolds with volume

In terms of the work of Gromov [Gro82] and Thurston [Thu79], the volume under hyperbolic metric is a topological invariant. More precisely, Gromov-Thurston theorem (see Theorem 2.3 in Section 2.) implies that the hyperbolic volume is propotional to simplicial volume. In the following, we briefly introduce known results so far on counting hyperbolic manifolds with volume.

Let C be a sufficiently large positive constant. Denote by $\rho_n(C)$ the number of hyperbolic manifolds of dimension at least four and volume at most C. Wang [Wan72] showed that $\rho_n(C)$ is finite. The first quantitative bound of $\rho_n(C)$ is the following upper bound given by Gromov in [Gro81],

$$\rho_n(C) \leqslant e^{e^{e^{C+n}}}.$$

The precise estimate of $\rho_n(C)$ occurs in [BGLM02]. Moreover, counting arithmetic hyperbolic manifolds is discussed in [BGLS10]. Gelander and Levit [GL14] counted the commensurability classes of hyperbolic manifolds.

The precise counting estimate of $\rho_n(C)$ is as follows.

Theorem 3.1 ([BGLM02])**.** *There exist two positive constants a_n and b_n only depending on n, such that*

$$a_n C \log C \leqslant \log \rho_n(C) \leqslant b_n C \log C. \tag{3.1}$$

Let $PO(n,1)$ be the isometry group of hyperbolic n-space $\mathbb{H}^n$. In [BGLM02], the proof of the lower bound of (3.1) is given by counting subgroups of a non-arithmetic lattice in $PO(n,1)$. Existence of such non-arithmetic lattices is proved in [GPS88]. Equivalently, the lower bound (3.1) is yielded by counting the number of covering hyperbolic manifolds of a fixed non-arithmetic hyperbolic manifold.

Proposition 3.2. *Let r be a sufficiently large positive integer. There exist positive constants m_1 and v_0, such that the number of hyperbolic manifolds with volume equal to $r \cdot v_0$ is at least $\frac{1}{m_1} r!$.*

Proof. The proof of this proposition can be found in [BGLM02]. For the purpose of completeness, we provide the main steps in the following.

Gromov [GPS88] constructed non-arithmetic lattices $\Lambda \subset PO(n,1)$. According to Lubotzky [Lub96], there exists a subgroup $\triangle$ of Λ, so that a surjective homomorphism from $\triangle$ to the free group F_2 of rank two exists. Moreover, Selberg's lemma implies that there exists a torsion free subgroup $\tilde{\triangle}$ of $\triangle$. Hence $M_0 = \mathbb{H}^n/\tilde{\triangle}$ is a complete hyperbolic manifold of dimension n. Denote by v_0 the volume of M_0.

Let $s_n(F_2)$ be the number of index n subgroups of the free group F_2. From [LS03, Chapter 2.], for every n, $s_n(F_2)$ satisfies

$$s_n(F_2) \geqslant n^{n/2}.$$

Therefore we get a counting estimate for the number of n-sheeted covering manifolds of M_0. Two n-sheeted covering manifolds of M_0 are isometric, if the corresponding subgroups are commensurable. Let m_1 be the size of commensurator of $\tilde{\triangle}$. Then the number of n-sheeted covering hyperbolic manifolds of M_0 has the lower bound $\frac{1}{m_1}n!$. Hence we get the lower bound estimate in (3.1). □

Remark 3.3. An alternative proof of the lower bound of $\rho_n(C)$ is given in [GL14].

Upper bound estimate in Theorem 3.1 is obtained by counting the 2-skeletons of simplicial complex homotopy equivalent to M, see [BGLM02].

Other related works include Young [You05] and Petri [Pet17] of counting hyperbolic manifolds with the diameter.

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4. Counting homotopy types of compact aspherical manifolds

In this section we consider the problem of counting homotopy types in terms of embolic volume and convex volume. We prove Theorem 1.2 in the following.

The proof of Theorem 1.2 is yielded by Theorem 1.4.

4.1. Proof of Theorem 1.2.

(1) **Lower bound**

All closed hyperbolic manifolds are aspherical. We give the lower bound in (1.2) by counting hyperbolic manifolds M satisfying $\mathsf{Emb}(M) \leqslant L$.

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In [Rez95], Reznikov proved that for a hyperbolic manifold M with dimension $n \geqslant 4$,

$$\mathsf{Inj}(M, \mathsf{hyp}) \geqslant C_n \, \mathsf{Vol}_{\mathsf{hyp}}(M)^{-\frac{n+1}{n-3}}, \tag{4.1}$$

where C_n is a positive constant only depending on n. For sufficiently large positive constant L, we consider hyperbolic n-manifolds M with

$$\mathsf{Vol}_{\mathsf{hyp}}(M) \leqslant C_n^{\frac{n(n-1)}{2n-2}} L^{\frac{n-3}{2n-2}}.$$

Then embolic volumes of these hyperbolic n-manifolds M satisfy

$$\mathsf{Emb}(M) \leqslant \frac{\mathsf{Vol}_{\mathsf{hyp}}(M)}{\mathsf{Inj}^n_{\mathsf{hyp}}(M)} \leqslant L. \tag{4.2}$$

According to (3.1), a lower bound for these hyperbolic manifolds can be established. Moreover, it is also the lower bound of all compact manifolds with embolic volume $\mathsf{Emb}(M) \leqslant L$. Hence

$$\log \varphi_n(L) \geqslant a_n L^{\frac{n-3}{2n-2}} \log L, \tag{4.3}$$

where a_n is a positive constant only depending on n.

In dimension 3, if we let injectivity radius be bounded by below, the method in [BGLM02] still produces the same amount of hyperbolic manifolds. Hence (4.3) also holds.

(2) **Upper bound**
Let $L > 0$ be a sufficiently large number. Set

$$\tilde{L} = \sup\{\mathsf{CV}(M) | \mathsf{Emb}(M) \leqslant L\},$$

then we have $\varphi_n(L) \leqslant \theta_n(\tilde{L})$, so that Theorem 1.4 implies that

$$\log \varphi_n(L) \leqslant b_n \tilde{L}^3 \log \tilde{L}.$$

□

Now we prove Theorem 1.4 in the following.

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4.2. **Proof of Theorem 1.4.** We separate the proof into two parts, according to lower bound and upper bound in (1.4) respectively.

(1) **Lower bound:**
We show that the number of closed hyperbolic manifolds with bounded convex volume $\mathsf{CV}(M)$ satisfies the lower bound in (1.4).
Since under hyperbolic metric hyp on M,

$$\mathsf{Conv}(M, \mathsf{hyp}) = \frac{1}{2} \mathsf{Inj}(M, \mathsf{hyp}),$$

we apply the same strategy as in the proof of Theorem 1.2. Then we use (3.1) to get

$$\log \theta_n(C) \geqslant c'_n C^{\frac{n+1}{n(n-1)}} \log C,$$

where c'_n is a positive constant only depending on n.

(2) **Upper bound:**
Let $(M, \mathcal{G})$ be a compact manifold of dimension n. Take $r = \frac{1}{10} \mathsf{Conv}(M, \mathcal{G})$. Let $\mathcal{U}_1 = \{B(p_j, r)\}$ be a maximal system of mutually disjoint metric balls with radius r in M. Then $\mathcal{U}_2 = \{B(p_j, 2r)\}$ is a covering to the manifold M. The nerve of covering $\mathcal{U}_2$ is denoted by $\mathcal{N}$. Each pair of balls in $\mathcal{U}_2$ has either empty intersection or convex intersection. Hence, according to nerve lemma (see Theorem 2.2), $\mathcal{N}$ is a simplicial complex homotopy equivalent to M. Denote by $\lambda_0(\mathcal{G})$ the number of vertices in the nerve $\mathcal{N}$.
For any ball $B(p_j, r) \in \mathcal{U}_1$, inequality (1.3) yields

$$\mathsf{Vol}_{\mathcal{G}}(B(p_j, r)) \geqslant C_n r^n.$$

We estimate upper bound of $\lambda_0(\mathcal{G})$ as follows,

$$\begin{aligned} \lambda_0(\mathcal{G}) &\leqslant \frac{\mathsf{Vol}_{\mathcal{G}}(M)}{\inf_j \mathsf{Vol}_{\mathcal{G}}(B(p_j, r))} \\ &\leqslant \frac{\mathsf{Vol}_{\mathcal{G}}(M)}{c_n r^n} \\ &= \frac{10^n}{c_n} \frac{\mathsf{Vol}_{\mathcal{G}}(M)}{\mathsf{Conv}(M, \mathcal{G})^n}. \end{aligned}$$

Denote by λ_0 the universal lower bound of $\lambda_0(\mathcal{G})$, i.e.,

$$\lambda_0 = \inf_{\mathcal{G}} \lambda_0(\mathcal{G}),$$

where the infimum is over all Riemannian metrics $\mathcal{G}$ on M. We have

$$\lambda_0 \leqslant \frac{10^n}{c_n} \mathsf{CV}(M).$$

Denote by $d(\mathcal{G})$ the upper bound of degree of vertices in the nerve $\mathcal{N}$. A very crude estimate of d is $\lambda_0(\mathcal{G})$. Let $\lambda_1(\mathcal{G})$ be the number of 1-skeletons of $\mathcal{N}$. Since the number of one dimensional simplicial complexes with at most $\lambda_0(\mathcal{G})$ vertices is bounded above by $\lambda_0(\mathcal{G})^{d(\mathcal{G})}$, we have

$$\begin{aligned}\lambda_1(\mathcal{G}) &\leqslant \lambda_0(\mathcal{G})^{d(\mathcal{G})\,\lambda_0(\mathcal{G})} \\ &= e^{d(\mathcal{G})\,\lambda_0(\mathcal{G})\log\lambda_0(\mathcal{G})}.\end{aligned}$$

Take the infimum over all Riemannian metrics $\mathcal{G}$ on M, then we know that the universal lower bound λ_1 of the number $\lambda_1(\mathcal{G})$ satisfies

$$\lambda_1 \leqslant e^{c_n'' \mathsf{CV}(M)^2 \log \mathsf{CV}(M)},$$

where c_n'' is a positive constant only depending on n.

Let $\lambda_2(\mathcal{G})$ be the number of 2-skeletons of $\mathcal{N}$. We have

$$\lambda_2(\mathcal{G}) \leqslant 2^{d^2\lambda_0(\mathcal{G})}\lambda_1(\mathcal{G}).$$

After taking infimum over all Riemannian metrics $\mathcal{G}$ on M, we obtain that the universal lower bound λ_2 satisfies

$$\lambda_2 \leqslant e^{\tilde{c}_n \mathsf{CV}(M)^3 \log \mathsf{CV}(M)},$$

where $\tilde{c}_n$ is a positive constant only depending on n.

Since the manifold M is aspherical, by Whitehead Theorem we know the homotopy type of M is determined by its fundamental group. Moreover, in a simplicial complex the 2-skeleton determines its fundamental group. Therefore, upper bound of λ_2 yields an upper bound of the number of homotopy types of M.

□

5. Counting hyperbolic manifolds with systolic volume

5.1. **Proof of Theorem 1.8.** Since hyperbolic 3-manifolds are different than the hyperbolic manifolds of dimension at least four, the proof is separated into two steps with respect to the dimension $n \geqslant 4$ or $n = 3$.

1. n ⩾ 4. Let M be a closed hyperbolic manifold of dimension $n \geqslant 4$. For a sufficiently large positive constant C, if $\mathsf{SR}(M) \leqslant C$, then according to Gromov's theorem of systolic volume (Theorem 2.4),

$$\begin{aligned}\mathsf{Vol}_{\mathsf{hyp}}(M) &= \mathcal{V}_n \|M\| \\ &\leqslant \mathcal{V}_n C_n \,\mathsf{SR}(M) \log^n \left(C_n' \,\mathsf{SR}(M)\right) \\ &< \tilde{C}_n C \log^n C,\end{aligned}$$

where $\tilde{C}_n = \mathcal{V}_n C_n$. Hence if we apply Theorem 3.1, the number $\rho_n(C)$ of closed hyperbolic manifolds M with $\mathsf{SR}(M) \leqslant C$ satisfies

$$a_n' C \log^{n+1} C \leqslant \log \rho_n(C) \leqslant b_n' C \log^{n+1} C,$$

where a_n' and b_n' are two positive constants only depending on n.

2. $\mathbf{n=3}$.
(1) Lower bound. In [BGLM02], the authors showed that there exists a hyperbolic manifold $M_0 = \mathbb{H}^n/\triangle$, where $\triangle$ is a discrete subgroup of $PO(n,1)$. Moreover, the hyperbolic n-manifold M_0 has at least $\frac{1}{m_1}r!$ r-sheeted mutually non-isometric covering n-manifolds, where m_1 and r are positive integers.

Note that

$$\mathsf{Sys}\,\pi_1(M_0,\mathsf{hyp}) \leqslant \mathsf{Sys}\,\pi_1(\tilde{M},\widetilde{\mathsf{hyp}}).$$

If we let $\delta_0 = \mathsf{Sys}\,\pi_1(M_0,\mathsf{hyp})$, then

$$\begin{aligned}\mathsf{SR}(\tilde{M}) &\leqslant \frac{\mathsf{Vol}_{\tilde{\mathsf{hyp}}}(\tilde{M})}{\mathsf{Sys}\,\pi_1(\tilde{M},\tilde{\mathsf{hyp}})}\\ &\leqslant \frac{r\cdot \mathsf{Vol}_{\mathsf{hyp}}(M_0)}{\mathsf{Sys}\,\pi_1(M_0,\mathsf{hyp})}\\ &= \frac{r}{\delta_0}\mathsf{Vol}_{\mathsf{hyp}}(M_0).\end{aligned}$$

Suppose that C is a sufficiently large positive constant, and $\frac{C}{2} \leqslant \frac{r}{\delta_0}\mathsf{Vol}_{\mathsf{hyp}}(M_0) \leqslant C$. Then the above inequality implies that $\mathsf{SR}(\tilde{M}) \leqslant C$. Hence, by using the construction of hyperbolic manifolds in [BGLM02], we obtain at least $a_n' C\log C$ hyperbolic 3-manifolds $\tilde{M}$ with systolic volume at most C.

(2) Upper bound
Let C be a sufficiently large positive constant. When M is a closed hyperbolic 3-manifold with $\mathsf{SR}(M) \leqslant C$, then according to Sabourau's theorem, there are only finitely many of them. Hence the injectivity radii of these closed hyperbolic 3-manifolds have a lower bound $A(C)$. According to Gromov's theorem of relating systolic volume with simplicial volume (Theorem 2.4) and the homotopy counting method from [BGLM02], we have

$$\log\rho_3(C) \leqslant A(C)\,C\log^{n+1}(C),$$

where $A(C)$ is a function of C only depending on C.

5.2. **Counting hyperbolic manifolds with embolic volume.** We show the proof of Propostion 1.10 here.

Proof of Propostion 1.10. Concerning embolic volume of a compact manifold M with nonzero simplicial volume $\|M\|$, Katz and Sabourau showed that

$$\|M\| \leqslant C_n\,\mathsf{Emb}(M)\log^n\left(C_n'\,\mathsf{Emb}(M)\right). \tag{5.1}$$

Let M be a compact hyperbolic manifold of dimension n. For a sufficiently large positive constant C, if $\mathsf{Emb}(M) \leqslant C$, then we can get a similar counting estimate in terms of (5.1). When the dimension $n \geqslant 4$, we get the following counting estimate by using the method of Burger, Gelander et al. [BGLM02],

$$a_n'' C\log^{n+1} C \leqslant \rho_n'(C) \leqslant b_n'' C\log^{n+1} C,$$

where a_n'' and b_n'' are two positive constants only depending on n. When the dimension $n=3$, $\mathsf{Emb}(M) \leqslant C$ leads to a lower bound of the injectivity radius $\mathsf{Inj}(M,\mathsf{hyp})$, since there are only finitely many compact hyperbolic 3-manifolds satisfying $\mathsf{Emb}(M) \leqslant C$. Upper bound of hyperbolic volume and lower bound of injectivity radius then result in the following counting estimate,

$$b\,C\log C \leqslant \rho_3'(C) \leqslant B(C)\,C\log^4(C),$$

where $B(C)$ is a function only depending on C.

□

5.3. **Counting hyperbolic manifolds using convex volume.** If we consider convex volume $\mathsf{CV}(M)$, we still get the similar counting result by using (5.2). Since $\mathsf{Emb}(M) \leqslant \mathsf{CV}(M)$, (5.1) implies

$$\|M\| \leqslant C_n\, \mathsf{CV}(M) \log^n \left(C_n'\, \mathsf{CV}(M)\right). \tag{5.2}$$

The above argument could be applied to get counting over the number $\rho_n''(C)$ of hyperbolic n-manifolds with convex volume at most C,

$$a_n'' C \log^{n+1} C \leqslant \rho_n''(C) \leqslant \begin{cases} b_n'' C \log^{n+1} C, & n \geqslant 4, \\ b(C)\, C \log^4 C, & n = 3. \end{cases}$$

where a_n'', b_n'' are positive constants only depending on n, and $b(C)$ a positive constant related to the given constant C.

6. Homotopy complexity and covering type

6.1. **Proof of Theorem 1.12.** Let $R = \frac{1}{2}\,\mathsf{Conv}(M, \mathcal{G})$. Cover M by a maximal system $\mathcal{U} = \{B(p_j, R)\}_{j=1}^{\delta}$ of disjoint metric balls with the common radius R. Hence if any metric ball of radius R is added into $\mathcal{U}$, there will be some intersections with the balls in $\mathcal{U}$. The manifold M is then covered by the collection $\widetilde{\mathcal{U}} = \{B(p_j, 2R)\}_{j=1}^{\delta}$ of balls with the same centers but doubled radius $2R$.

Croke's local embolic inequality (see [Cro80, Proposition 14]) implies that for any point $p \in M$,

$$\mathsf{Vol}_{\mathcal{G}}(B(p, r)) \geqslant \alpha_n r^n \tag{6.1}$$

holds if $0 < r \leqslant \frac{1}{2}\,\mathsf{Inj}(M, \mathcal{G})$, where α_n is a positive constant only depending on n. Since $\mathsf{Inj}(M, \mathcal{G}) \geqslant 2\,\mathsf{Conv}(M, \mathcal{G})$ (see [Ber76] for a proof), (6.1) yields the following local inequality of convex radius.

Lemma 6.1. *For any Riemannian metric $\mathcal{G}$ defined on a compact manifold M of dimension n,*

$$\mathsf{Vol}_{\mathcal{G}}(B(p, r)) \geqslant \alpha_n r^n \tag{6.2}$$

holds for any point $p \in M$ if $r \leqslant \mathsf{Conv}(M, \mathcal{G})$, where α_n is a constant only depending n.

By Lemma 6.1,

$$\begin{aligned} \mathsf{Vol}_{\mathcal{G}}(M) &\geqslant \sum_{j=1}^{\delta} \mathsf{Vol}_{\mathcal{G}}(B(p_j, R)) \\ &\geqslant \sum_{j=1}^{\delta} \alpha_n R^n \\ &= \delta\, \alpha_n R^n. \end{aligned}$$

Hence we have

$$\delta \leqslant \frac{\mathsf{Vol}_{\mathcal{G}}(M)}{\alpha_n R^n}.$$

After taking infimum over all Riemannian metrics $\mathcal{G}$ on M, we have

$$\delta \leqslant \frac{2^n}{\alpha_n}\,\mathsf{CV}(M).$$

Let $\mathcal{N}$ be the nerve of the covering $\widetilde{U} = \{B(p_j, 2R)\}_{j=1}^{\delta}$. The number of vertices in $\mathcal{N}$ is equal to the number δ of balls in $\widetilde{\mathcal{U}}$. Moreover, any metric ball of radius $R = \frac{1}{2}\,\mathsf{Conv}(M, \mathcal{G})$ is geodesically convex. Hence $\widetilde{\mathcal{U}}$ is a good cover (see Definition 2.1). In terms of the nerve theorem (Theorem 2.2), we know that $\mathcal{N}$ is homotopy equivalent to M.

If two balls $B(p_i, 2R)$ and $B(p_j, 2R)$ have non empty intersections, then

$$B(p_i, R) \subset B(p_j, 5R).$$

Therefore, an upper bound of the number of intersections for a given ball $B(p_j, 2R)$ is

$$D = \frac{\mathsf{Vol}_{\mathcal{G}}(B(p_j, 5R))}{\inf_j \mathsf{Vol}_{\mathcal{G}}(B(p_i, R))},$$

where the infimum is taken over all balls $B(p_i, R)$ such that $B(p_i, 2R)$ having nonempty intersections with $B(p_j, 2R)$. Apply inequality (6.2), an estimate for the upper bound D is $\frac{2^n}{\alpha_n}\,\mathsf{CV}(M)$.

6.2. **Proof of Corollary 1.14.** In the above proof of Theorem 1.12, a nerve $\mathcal{N}$ homotopy equivalent to the compact manifold M is constructed. Moreover, the number δ of the vertices of $\mathcal{N}$ satisfies

$$\delta \leqslant \frac{2^n}{\alpha_n}\,\mathsf{CV}(M).$$

Let $C_n = \frac{2^n}{\alpha_n}$. Then we get the inequality (1.7).

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